Tail asymptotic of the stationary distribution for the state dependent (1,R)-reflecting random walk: near critical[1]

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Abstract

In this paper, we consider the $(1, R)$ state-dependent reflecting random walk (RW) on the half line, allowing the size of jumps to the right at maximal R and to the left only 1. We provide an explicit criterion for positive recurrence and the explicit expression of the stationary distribution based on the intrinsic branching structure within the walk. As an application, we obtain the tail asymptotic for the stationary distribution in the "near critical" situation.

Keywords: random walk, multi-type branching process, positive recurrence, stationary distribution, tail asymptotic.
Mathematics Subject Classification: Primary 60K37; Secondary 60J85

1 Introduction and Main Results

1.1 The background and motivation

We consider the $(1, R)$-reflecting random walk on the half line, i.e., a Markov chain $\{X_m\}_{m\geq 0}$ on $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ with $X_0 = 0$ and the transition probabilities P_{ij} specified by for $i \geq 0$ $(q(0) = 0)$,

$$P_{ij} = \begin{cases} r(i), & \text{for } j = i, \\ q(i), & \text{for } j = i - 1, \\ p_{j-i}(i), & \text{for } i < j \leq i + R, \\ 0, & \text{otherwise,} \end{cases}$$

where $r(i) + q(i) + p_1(i) + p_2(i) + \cdots + p_R(i) = 1$, $0 < q(i) < 1$, for $i \geq 1$, and $r(i) \geq 0, p_1(i), p_2(i), \cdots, p_R(i) \geq 0$. Obviously, this Markov chain is irreducible. It can also be written as the transition matrix (for simplicity, $R = 2$),

$$\begin{pmatrix} r(0) & p_1(0) & p_2(0) & & & & \\ q(1) & r(1) & p_1(1) & p_2(1) & & & \\ & q(2) & r(2) & p_1(2) & p_2(2) & & \\ & & q(3) & r(3) & p_1(3) & p_2(3) & \\ & & \ddots & \ddots & \ddots & \ddots & \end{pmatrix}.$$

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in which all unspecified entries are zero.

For simplicity, we will restrict ourselves to consider $R = 2$, and we write the transition probability at position i as $P(i) = (q(i), r(i), p_1(i), p_2(i))$ (recall $q(0) = 0$ and $0 < q(i) < 1$). At first, if the transition probability of the $(1,2)$-RW $(X_m)_{m\geq 0}$ is *state independent*, i.e., $P(i) \equiv P = (q, r, p_1, p_2)$ for $i \geq 1$. Let (see figure 1)

$$D = \{(q, r, p_1, p_2) : p_1 + p_2 + q + r = 1; \quad p_1 + 2p_2 < q\},$$
$$L = \{(q, r, p_1, p_2) : p_1 + p_2 + q + r = 1; \quad p_1 + 2p_2 = q\},$$

it is easy to see that $(X_m)_{m\geq 0}$ is positive recurrent iff $P(i) \equiv P = (q, r, p_1, p_2) \in D$ $(i \geq 1)$ and null recurrent iff $P(i) \equiv P = (q, r, p_1, p_2) \in L$ $(i \geq 1)$.

How about the situation for the *state-dependent* $(1, R)$-RW $(X_m)_{m\geq 0}$? To our best knowledge only for $R = 1$, i.e., *state-dependent* $(1,1)$-RW, the criteria for the (positive) recurrence and the expression for the stationary distribution have been given explicitly (see for example [12] and [13]), and further tail asymptotic for the stationary distribution have been found in [4] (Page 294 and Page 305).

The aim of the present paper is to give an explicit criteria of the positive recurrence and explicit expressions of the stationary distribution for the *state-dependent* $(1, R)$-RW, which enable us to consider the tail asymptotic of the stationary distribution. Our method is based on the intrinsic branching structure within the random walk ([8], [9]).

1.2 Main results

1.2.1 *Criteria for the positive recurrence and stationary distribution*

Define

$$\alpha_k = p_k(0) + p_{k+1}(0) + \cdots + p_R(0), \text{ for } 1 \leq k \leq R,$$
$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_R), \; e_1 = (1, 0, \cdots, 0),$$
$$\theta_k(i) = \frac{p_k(i) + p_{k+1}(i) + \cdots + p_{R-1}(i) + p_R(i)}{q(i)},$$
$$M_i = \begin{pmatrix} \theta_1(i) & \theta_2(i) & \dots & \theta_{R-1}(i) & \theta_R(i) \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{R\times R}, \tag{1.1}$$

Theorem 1.1. *Assume for $i \geq 0$,*

$$\mu(0) = 1; \qquad \mu_1 = \frac{1}{q(1)}\alpha e_1';$$
$$\mu(i) = \frac{1}{q(i)}\alpha M_1 M_2 \cdots M_{i-1} e_1'. \tag{1.2}$$

(i) $\mu(i)$ $(i \geq 0)$ are the stationary measure of the state-dependent $(1, R)$-RW $(X_m)_{m\geq 0}$.

(ii) If $\sum_{i=0}^{\infty} \mu(i) < \infty$, then the walk $\{X_m\}_{m\geq 0}$ is positive recurrence. Furthermore the stationary distribution can be expressed as

$$\pi(i) = \frac{\mu(i)}{\sum_{i=0}^{\infty} \mu(i)}. \tag{1.3}$$

□

Remark (1.2) generalize the classical results for the *state-dependent* $(1,1)$-RW, see for example [4] (Page 297). □

1.2.2 *Tail asymptotic of the stationary distribution: near critical*

With the explicit expression of the stationary distribution (1.3) at hand, we can consider the tail asymptotic of the distribution. Firstly, it is not difficult (but is also not obviously, as [10] for the $(L,1)$-RW) to see that the tail of $\pi(i)$ is geometric decay in the sense $\lim_{i\to\infty}\frac{\log\pi(i)}{i}=-c<0$ when the transition probability $P(i)\to P=(q,r,p_1,p_2)\in D$. What we are now interested in is the "near critical" situation: the transition probability $P(i)$ from the interior of the "positive recurrence area D" to P in the "null recurrence area L" as $i\to\infty$. See figure 1 (In this figure, we assume r=0).

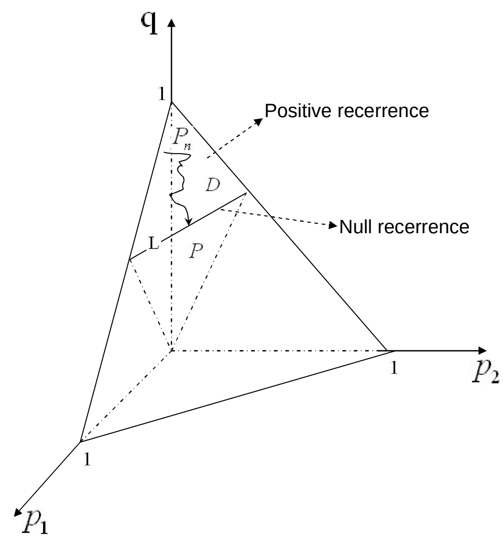

figure 1: District of the transition probability

One of the interesting phenomena is that even all the $P(i)\in D$, the "positive recurrence area", the walk X_m could be null recurrent. To this end, we need to consider a finer manner of the $P(i)$ goes to P as $i\to\infty$. Let $P=(p_1,p_2,r,q)\in L$, and for $i\geq 1$, $P(i)=(q(i),r(i),p_1(i),p_2(i))$ is given by

$$\begin{aligned}&p_1(i)=p_1-\varepsilon_i,\ p_2(i)=p_2-\varepsilon_i,\ q(i)=q+\varepsilon_i;\\&r(i)=1-p_1(i)-p_2(i)-q(i).\end{aligned}\tag{1.4}$$

where $\varepsilon_i>0$ and small enough, $\varepsilon_i\downarrow 0$ as $i\to\infty$. It is obvious that $P(i)\in D$, $P\in L$, and $P(i)\to P$ as $i\to\infty$.

Theorem 1.2. *(a) If $\sum_{i=0}^{\infty}\varepsilon_i<\infty$, X_m is null recurrence.*

(b) if $\sum_{i=0}^{\infty}\varepsilon_i^p<\infty$ for some $1<p\leq 2$, $\kappa=\frac{4}{q}$.

(b1) When $\sum_{i=0}^{\infty}\prod_{k=0}^{i}e^{-\kappa\varepsilon_k}<\infty$, X_m is positive recurrence, and

$$\log\pi(i)\sim-\kappa\sum_{k=0}^{i}\varepsilon_k,\quad \textit{as } i\to\infty.$$

(b2) When $\sum_{i=0}^{\infty}\prod_{k=0}^{i}e^{-\kappa\varepsilon_k}=\infty$, X_m *is null recurrence.* □

As an application, we immediately have the following

Corollary 1.1. *Suppose* $\varepsilon_i \sim Ci^{-\alpha}$ *as* $i\to\infty$, C *is a positive constant.*
$Case\ 1:$ $\alpha>1$, X_m *is null recurrence.*
$Case\ 2:$ $\frac{1}{2}<\alpha<1$, X_m *is positive recurrence, and we have*

$$\log\pi(i)\sim-\frac{C\kappa}{1-\alpha}i^{1-\alpha},\quad \textit{as } i\to\infty.$$

$Case\ 3:$ $\alpha=1$, *If* $C\kappa<1$, X_m *is null recurrence; if* $C\kappa>1$, X_m *is positive recurrence, and*

$$\log\pi(i)\sim-C\kappa\log i,\quad \textit{as } i\to\infty.$$

□

Remark Theorem 1.2 and Corollary 1.1 say that even all the $P(i)\in D$, the walk X_m could be null recurrent, which generalize the results for the *state-dependent* $(1,1)$-RW ([4], Page 294 and Page 305). □

We arrange the remainder of this paper as follows. In Section 2, we will prove Theorem 1.1 after a brief review about the intrinsic branching structure within the walk, which is the basic tool to specify the stationary measure; Theorem 1.2 and Corollary 1.1 will be proved in Section 3.1, together with some preparations on the asymptotic solution of difference system.

2 Proof of Theorem 1.1

Let $T=\inf\{m>0: X_m=0\}$, $N(i)=\sum_{m=0}^{T-1}1_{\{X_m=i\}}$ be the number of visits to state i by the chain before T, and E^i is the expectation when the walk starts at $X_0=i$. Firstly, recall a classical results on the (positive) recurrence and the stationary distribution of a general Markov chain X_m.

Proposition 2.1. *(Thoerem (4.3), [4]) For* $k\geq 0$, $\mu(i)=E^0N(i)$ *defines a stationary measure. If* $\sum_{k=0}^{\infty}\mu(i)<\infty$, *the random walk is positive recurrence, and the stationary distribution can be expressed as*

$$\pi(i)=\frac{\mu(i)}{\sum_{i=0}^{\infty}\mu(i)}.$$

□

We can calculate the $\mu(i)=E^0N(i)$ by the intrinsic branching structure within the $(1,R)$-RW as follows (the proof will delay at the end of this section),

Proposition 2.2. *We have* $E^0N(0)=1$, *and*

$$E^0N(1)=\frac{1}{q(i)}\alpha e_1'=\frac{p_1(0)+p_2(0)}{q(1)},$$
$$E^0N(i)=\frac{1}{q(i)}\alpha M_1M_2\cdots M_{i-1}e_1',\quad \textit{for}\ \ i>1,$$

where α, M_i are given in (1.1). $\square$

Proof of Theorem 1.1. With Proposition 2.1 and 2.1 at hand, Theorem 1.1 is immediately. $\square$

What we should do is to prove Proposition 2.2, our method is the intrinsic branching structure within the $(1, R)$-RW ([8], 2009).

Brief review for the intrinsic branching structure. The intrinsic branching structure within a random walk has been studied by many authors. For the $(1,1)$-RW, Dwass ([5], 1975) and Kesten *et al.* ([14], 1975) observed a Galton-Watson process with the geometric offspring distribution hidden in the nearest random walk. The branching structure is a powerful tool in the study of random walks in a random environment (RWRE, for short). In [14], Kesten *et al.*, proved a stable law for the nearest RWRE by using this branching structure. The key point is that the hitting time T_i can be calculated accurately by the branching structure.

However, if the random walk is allowed to jump even to a bounded range, referred to as the (L, R)-RW, the situation will become much more complicated. A multi-type branching process has been revealed by Hong & Wang ([8], 2009) for the $(L, 1)$-RW, and a little bit late for the $(1, R)$-RW ([9], 2010) by Hong & Zhang. It must be emphasized that these two branching structures are not symmetric, instead they are essentially different. Note that if we assume $q_2(i) \equiv 0$, both branching structures degenerate to the case of the $(1,1)$-RW.

The following discussion is based on $R = 2$. The general case can be similarly discussed, but the notation is much more complicated. Assume that $X_0 = 0$ we can calculate $E^0 N_i$ by using the branching structure within the random walk ([8], 2009). Note that we consider the reflected $(1, R)$-RW and calculate $E^0 N_i$ before first return the start position 0, actually we use the branching structure for the $(R, 1)$-RW by Hong & Wang ([8], 2009), and with a little modification because of considering the walk could be stay at each state i (here $r(i) \geq 0$).

Recall that $T = \inf\{n > 0, X_n = 0\}$, define

$$\begin{aligned} U_k^m &= \#\{0 \leq j < T : X_j \leq k,\ X_{j+1} = k + m\} \quad \text{ for } k \geq 0,\ m = 1, 2. \\ U_k^3 &= \#\{0 \leq j < T :\ X_j = k, X_{j+1} = k\} \quad \text{ for } k \geq 0, \end{aligned}$$

Setting

$$U_k = (U_k^1, U_k^2, U_k^3) \quad \text{ for } k \geq 0.$$

We then have the following property (with a little modification)

Theorem A (Hong and Wang [8]) *(1) The process* $\{U_n\}_{n=0}^{\infty}$ is a 3-type branching process whose branching mechanism is given by,

$$\begin{aligned} P(U_0 = (1,0,0)) &= p_1(0), \\ P(U_0 = (0,1,0)) &= p_2(0), \\ P(U_0 = (0,0,1)) &= r(0); \end{aligned} \tag{2.1}$$

and for $k \geq 0$

$$P(U_{k+1} = (a,b,c) \big| U_k = e_1) = \frac{(a+b+c)!}{a!b!c!} r(k)^a p_1(k)^b p_2(k)^c q(k),$$

$$P(U_{k+1}=(a,1+b,c)\big|U_k=e_2)=\frac{(a+b+c)!}{a!b!c!}r(k)^a p_1(k)^b p_2(k)^c q(k),$$
$$P(U_{k+1}=(0,0,0)\big|U_k=e_3)=1.$$

(2) For the process $\{U_n\}_{n=0}^{\infty}$, let $\widetilde{M}_k$ be the 3×3 mean matrix whose m-th row is $E(U_{k+1}|U_k=e_m)$, for $k\geq 0$. Then, one has that

$$\widetilde{M}_k=\begin{pmatrix} \frac{p_1(k)}{q(k)} & \frac{p_2(k)}{q(k)} & \frac{r(k)}{q(k)} \\ 1+\frac{p_1(k)}{q(k)} & \frac{p_2(k)}{q(k)} & \frac{r(k)}{q(k)} \\ 0 & 0 & 0 \end{pmatrix}, \quad k\geq 1.$$

□

Now we are at the position to prove Proposition 2.2.

Proof of Proposition 2.2 It is not hard to deduce the relationship between the random walk and the intrinsic branching structure that $E^0N(0)=1$, and for $i\geq 1$, $N(i)=U_{i-1}^1+|U_i|$ (where$|U_i|=U_i^1+U_i^2+U_i^3$).

$$\begin{aligned} E^0N(1)&=p_1(0)+E^0U_0\widetilde{M}_1(1,1,1)' \\ &=p_1(0)+E^0U_0(\frac{1}{q(1)}-1,\frac{1}{q(1)},0)' \\ &=p_1(0)+p_1(0)(\frac{1}{q(1)}-1)+p_2(0)\frac{1}{q(1)} \\ &=\frac{p_1(0)+p_2(0)}{q(1)}. \end{aligned}$$

For $i>1$, using the Markov property, we have

$$E^0(N(i)\big|U_{i-1},U_{i-2},...,U_0)=U_{i-1}^1+|U_{i-1}\widetilde{M}_i|.$$

As a consequence

$$\begin{aligned} E^0N(i)&=E^0U_{i-1}e_1'+E^0U_{i-1}\widetilde{M}_i(1,1,1)' \\ &=E^0U_{i-2}\widetilde{M}_{i-1}e_1'+E^0U_{i-2}\widetilde{M}_{i-1}\widetilde{M}_i(1,1,1)' \\ &=E^0U_0\widetilde{M}_1\widetilde{M}_2\cdots\widetilde{M}_{i-1}e_1'+E^0U_0\widetilde{M}_1\widetilde{M}_2\cdots\widetilde{M}_{i-1}\widetilde{M}_i(1,1,1)'. \end{aligned}$$

By (2.1), $E^0U_0=(p_1(0),p_2(0),r(0)):=\beta$,

$$\begin{aligned} E^0N(i)&=\beta\widetilde{M}_1\widetilde{M}_2\cdots\widetilde{M}_{i-1}e_1'+\beta\widetilde{M}_1\widetilde{M}_2\cdots\widetilde{M}_{i-1}\widetilde{M}_i(1,1,1)' \\ &=\frac{1}{q(i)}\beta\widetilde{M}_1\widetilde{M}_2\cdots\widetilde{M}_{i-1}(1,1,0)'. \end{aligned} \tag{2.2}$$

Define

$$\widehat{M}_i=\begin{pmatrix} \frac{p_1(i)+p_2(i)}{q(i)} & \frac{p_2(i)}{q(i)} & \frac{r(i)}{q(i)} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

notice that

$$\widetilde{M_k} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \widehat{M_k} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}.$$

Substitute the above equation into (2.2), by some calculations

$$\begin{aligned} E^0N(i) &= \frac{1}{q(i)}\beta \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \widehat{M_1}\widehat{M_2}\cdots\widehat{M_{i-1}} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} (1,1,0)' \\ &= \frac{1}{q(i)}\beta \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M_2 & 0 \\ 0 & 0 \end{pmatrix} \cdots \begin{pmatrix} M_{i-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} (1,1,0)' \\ &= \frac{1}{q(i)}(p_1(0)+p_2(0), p_2(0))M_1M_2\cdots M_{i-1}(1,0)', \end{aligned}$$

complete the proof. □

3 Proof of Theorem 1.2

3.1 The Asymptotic Solution of Difference system

In this section, we introduce the asymptotic behavior of linear difference system. Here, we just consider the second-order system.

$$y_{k+1} = [\Lambda + R_k]y_k \quad k \geq 0 \tag{3.1}$$

where $y_k \in \mathbb{R}^2$, $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2\}$, and R_n is a small perturbation in a sense to be made precise. We assume that $|\lambda_1| > |\lambda_2| > 0$, and $\|R_n\| = \sum_{i=1}^n \sum_{j=1}^n |r_{ij}|$.

The classical result for asymptotic analysis of solutions is to represent a fundamental matrix in the form

$$Y_k = [I + o(1)] \prod_{i=0}^{k-1} \tilde{\Lambda}_i$$

where $\tilde{\Lambda}(l)$ is an explicitly diagonal matrix whose main terms come from Λ.

If we consider the difference equations $y_{k+1} = A_k y_k$. Use this asymptotic representation, we can give a precise estimation of the non-homogeneous matrix products $A_n A_{n-1} \cdots A_{k_0}$.

Here, we just give the case that R_n in (3.1) is an l^1-perturbations and l^p-perturbations with $1 < p \leq 2$. If $p > 2$, the result is more complicate.

3.2 l^1-perturbations

The fundamental theorem of Levinson ([15], 1948) establish analogous results for perturbed systems of differential equations. Benzaid and Lutz ([1], 1987) give the discrete analogue for difference equations. This theorem consider the more general case when Λ is depends on k, requiring a dichotomy condition on Λ and a growth condition on the perturbation R_k.

Proposition 3.1. *(Theorem 2.2 Benzaid and Lutz (P202)) Consider $y_{k+1} = [\Lambda + R_k]y_k$, where $\Lambda = diag\{\lambda_1, \lambda_2 \cdots \lambda_n\}$, $\lambda_i \neq 0$ for all $1 \leq i \leq n$, and $\sum_{k=k_0}^{\infty} \|R_k\| < \infty$. Then the system has a fundamental matrix satisfying, as $k \to \infty$*

$$Y_k = [I + o(1)]\Lambda^k. \tag{3.2}$$

3.3 l^p-perturbations with $1 < p \leq 2$

While the discrete version of Levinson's theorem considered l^1-perturbations R in (3.1), the discrete version of the theorem of Hartman-Wintner ([7], 1955)was concerned with l^p perturbations for some $1 < p \leq 2$. The proof is based on the so-called Q-transformation which was first introduced for differential equations by Harris and Lutz ([6], 1974) and later on modified for difference equations by Benzaid and Lutz ([1], 1987). Those methods have been well-established.

Proposition 3.2. *(Corollary 3.4 Benzaid and Lutz (P210)) Consider $y_{k+1} = [\Lambda + R_k]y_k$, where $\Lambda = diag\{\lambda_1, \lambda_2 \cdots \lambda_n\}$, $|\lambda_1| > |\lambda_2| > \cdots |\lambda_n| > 0$, and $\sum_{k_0}^{\infty} \|R_k\|^p < \infty$ for some $1 < p \leq 2$. Then the system has a fundamental matrix satisfying, as $k \to \infty$*

$$Y_k = [I + o(1)] \prod_{i=0}^{k-1} [\Lambda + \mathrm{diag} R_i] \tag{3.3}$$

3.4 From "positive recurrence area" to the boundary of null recurrence

In this section, we just consider when $R = 2$, and assume that $p_2(i), p_2 > 0$. The key to prove Theorem 1.2 is to discuss when $P(n) \to P$, the asymptotic representation of $M_1 M_2 \cdots M_n$. To this end, we consider the following difference system.

$$y_{n+1} = M_n' y_n = [M' + R_n'] y_n \tag{3.4}$$

where $R_n = M_n - M$,

$$M = \begin{pmatrix} \frac{p_1+p_2}{q} & \frac{p_2}{q} \\ 1 & 0 \end{pmatrix}, \quad M_n = \begin{pmatrix} \frac{p_1(n)+p_2(n)}{q(n)} & \frac{p_2(n)}{q(n)} \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that $1 = \lambda_1 > 0 > \lambda_2 > -1$, where λ_1, λ_2 are two eigenvalues of M. So it can be expressed in the diagonal form

$$T^{-1} M T = \mathrm{diag}(\lambda_1, \lambda_2)$$

where T is the non-singular matrix

$$T = \begin{pmatrix} 1 & \lambda_2 \\ 1 & 1 \end{pmatrix}.$$

Let $y_n = (T^{-1})' z_n$, and (3.4) becomes

$$z_{n+1} = [\mathrm{diag}(\lambda_1, \lambda_2) + T' R_n' (T^{-1})'] z_n. \tag{3.5}$$

Lemma 3.1. *Under our condition, we have for some constant* K_1, K_2*,*

$$K_1\varepsilon_n \le \|T'R_n'(T^{-1})'\| \le K_2\varepsilon_n, \quad as \;\; \varepsilon_n \to 0.$$

Proof. For fixed T, by [11], P295, Theorem 5.6.7. $\|T^{-1}\cdot T\|_\diamond = \|\cdot\|$ is also a norm of the matrix. Then by the equivalence of the norm, there exists constants $c_1,\ c_2$,

$$c_1\|R_n\| \le \|R_n\|_\diamond \le c_2\|R_n\|.$$

We can see

$$\|R_n\| = |\frac{p_1(n)+p_2(n)}{q(n)} - \frac{p_1+p_2}{q}| + |\frac{p_2(n)}{q(n)} - \frac{p_2}{q}|.$$

Using $p_1(n) \sim p_1 - C\varepsilon_n,\ p_2(n) \sim p_2 - C\varepsilon_n,\ q(n) \sim q + C\varepsilon_n$, we have for some constant C', $\|R_n\| \sim C'\varepsilon_n$. So there exist K_1, K_2 satisfy

$$K_1\varepsilon_n \le \|T'R_n'(T^{-1})'\| \le K_2\varepsilon_n$$

□

Combine (3.4), (3.5), Lemma 3.1 and the discuss above, we deduce from the Propositions 3.1 and 3.2 that,

Lemma 3.2. *If* $\sum_{n=0}^{\infty}\varepsilon_n < \infty$*, Then the system (3.4) has a fundamental matrix satisfying, as* $k \to \infty$

$$Y_k = (T^{-1})'[I+o(1)]\,diag(1, \lambda_2^k). \tag{3.6}$$

Proof. By Lemma 3.1, if $\sum_{n=0}^{\infty}\varepsilon_n < \infty$, we use Proposition 3.1 to system (3.5), we have

$$Z_k = [I+o(1)]\mathrm{diag}(1, \lambda_2^k). \tag{3.7}$$

Substitute $y_n = (T^{-1})'z_n$ we obtain (3.6), complete the proof. □

Lemma 3.3. *If for some* p *such that* $1 < p \le 2$, $\sum_{n=0}^{\infty}\varepsilon_n^p < \infty$*. Then the system (3.4) has a fundamental matrix satisfying, as* $k \to \infty$

$$Y_k = (T^{-1})'[I+o(1)]\prod_{i=0}^{k-1}[diag T^{-1}M_iT] \tag{3.8}$$

Proof. The proof is the same as Lemma 5.2,

$$\begin{aligned}\mathrm{diag}(\lambda_1,\lambda_2) + \mathrm{diag}T'R_n'(T^{-1})' &= \mathrm{diag}(\lambda_1,\lambda_2) + \mathrm{diag}T'M_n'(T^{-1})' - \mathrm{diag}T'M'(T^{-1})' \\ &= \mathrm{diag}T'M_n'(T^{-1})' = \mathrm{diag}T^{-1}M_nT\end{aligned}$$

□

3.5 Proof of Theorem 1.2

Proof. It is evident that $\widetilde{Y}_i := M_i' M_{i-1}' \cdots M_1'$ is a fundamental matrix of (3.4); on the other hand Y_i in (3.6) and (3.8) is also the fundamental matrix of (3.4). So there exists a nonsingular matrix $\widetilde{C}$ such that $\widetilde{Y}_i = Y_i \cdot \widetilde{C}$.

(a) If $\sum_{n=0}^{\infty} \varepsilon_n < \infty$.

Combine Theorem 1.1, Lemma 3.2 and (3.4),

$$\begin{aligned}
\mu(i) &= \frac{1}{q(i)} \alpha M_1 M_2 \cdots M_{i-1} e_1' = \frac{1}{q(i)} e_1 M_{i-1}' M_{i-2}' \cdots M_1' \alpha' \\
&= \frac{1}{q(i)} e_1 Y_{i-1} \widetilde{C} \alpha' \\
&= \frac{1}{q(i)} e_1 (T^{-1})' [I + o(1)] \mathrm{diag}(1, \lambda_2^{i-1}) \widetilde{C} \alpha'.
\end{aligned}$$

Because $|\lambda_2| < 1$, $\mu(i)$ has a positive limit as $i \to \infty$. So $\sum_{i=0}^{\infty} \mu(i) = \infty$, and X_n is null recurrence (It is evident that X_n is recurrent because all the transition probability P_i is in the "positive recurrent area D") .

(b) If $\sum_{n=0}^{\infty} \varepsilon_n = \infty$, and for some p such that $1 < p \le 2$, $\sum_{n=0}^{\infty} \varepsilon_n^p < \infty$. We can calculate $\mathrm{diag} T^{-1} M_i T$,

$$\begin{aligned}
\mathrm{diag} T^{-1} M_i T &= \frac{1}{1-\lambda_2} \begin{pmatrix} \frac{p_1(n)+2p_2(n)}{q(n)} - \lambda_2 & 0 \\ 0 & \lambda_2 (1 - \frac{p_1(n)+2p_2(n)}{q(n)}) - \frac{p_2(n)}{q(n)} \end{pmatrix} \\
&\sim \frac{1}{1-\lambda_2} \begin{pmatrix} 1 - \kappa\varepsilon_n - \lambda_2 & 0 \\ 0 & -\lambda_2 \kappa \varepsilon_n - \frac{p_2(n)}{q(n)} \end{pmatrix}.
\end{aligned}$$

By Lemma 3.3, use the same method, there exist an constant c, such that

$$\mu(i) \sim \frac{c}{(1-\lambda_2) q(i)} \prod_{k=0}^{i-1} (1 - \kappa \varepsilon_k).$$

Using the fact that $\log(1-\alpha) \sim -\alpha$ as $\alpha \to 0$ we see that

$$\log \prod_{k=0}^{i-1} (1 - \kappa\varepsilon_k) \sim -\sum_{k=0}^{i-1} \kappa \varepsilon_i \quad \text{as } k \to \infty,$$

then

$$\mu(i) \sim \frac{c}{(1-\lambda_2) q(i)} e^{-\sum_{k=0}^{i-1} \kappa\varepsilon_k}. \tag{3.9}$$

Note that $\frac{c}{(1-\lambda_2)q(i)}$ is a bounded sequence. If $\sum_{i=0}^{\infty} \prod_{k=0}^{i} e^{-\kappa\varepsilon_k} < \infty$,by (3.9), $\sum_{i=0}^{\infty} \mu(i) < \infty$, the process is positive recurrence. If $\sum_{i=0}^{\infty} \prod_{k=0}^{i} e^{-\kappa\varepsilon_k} = \infty$, by (3.9), $\sum_{i=0}^{\infty} \mu(i) = \infty$, it is null recurrence. when the process is positive recurrence,

$$\log \pi(i) = \log \frac{\mu(i)}{\sum_{i=0}^{\infty} \mu(i)} \sim -\kappa \sum_{k=0}^{i} \varepsilon_k. \tag{3.10}$$

□

3.6 Proof of Corollary 1.1

In this section, we consider the example to explain the boundary between null recurrence and positive recurrence. For the case $R = 1$, it can be found in ([4]).

Recall that $\kappa = \frac{4}{q}$. We assume $p_1(0) = p_1 - Cn^{-\alpha}$, $p_2(0) = p_2 - Cn^{-\alpha}$, and for $n > 0$, $p_1(n) = p_1 - Cn^{-\alpha}$, $p_2(n) = p_2 - Cn^{-\alpha}$, $q(n) = q + Cn^{-\alpha}$, where p_1, p_2, q satisfy $p_1 + 2p_2 = q$.
$Case\ 1:$ $\alpha > 1$. We have $\sum_{n=0}^{\infty} \varepsilon_n < \infty$. By Theorem 1.2 (a), X_n is null recurrence.
$Case\ 2:$ $\frac{1}{2} < \alpha < 1$. There exist $1 < p \leq 2$ such that $\sum_{n=0}^{\infty} \varepsilon_n^p < \infty$. And

$$\prod_{k=0}^{i} e^{-\kappa\varepsilon_k} \sim e^{-\frac{C\kappa}{1-\alpha} i^{1-\alpha}}.$$

So $\sum_{i=0}^{\infty} \prod_{k=0}^{i} e^{-\kappa\varepsilon_k} < \infty$. By Theorem 1.2 (b), the process is positive recurrence, and we have

$$\log \pi(i) \sim -\frac{C\kappa}{1-\alpha} i^{1-\alpha}, \quad \text{as } i \to \infty.$$

$Case\ 3:$ $\alpha = 1$. It is easy to see that $\sum_{n=0}^{\infty} \varepsilon_n^2 < \infty$. And

$$\prod_{k=0}^{i} e^{-\kappa\varepsilon_k} \sim e^{-C\kappa \log i} = i^{-C\kappa}.$$

So if $C\kappa < 1$, $\sum_{i=0}^{\infty} \prod_{k=0}^{i} e^{-\kappa\varepsilon_k} = \infty$ X_n is null recurrence; if $C\kappa > 1$, $\sum_{i=0}^{\infty} \prod_{k=0}^{i} e^{-\kappa\varepsilon_k} < \infty$, X_n is positive recurrence, and

$$\log \pi(i) \sim -C\kappa \log i, \quad \text{as } i \to \infty.$$

□

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